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New methods of the mass and heat transfer theory—II. The methods of asymptotic interpolation and extrapolation

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Abstract—A method for the construction of approximate formulae based on the use of the well-known two-sided asymptotic expressions of the solution to the corresponding problem (the method of asymptotic interpolation) is suggested as well as the technique for a substantial extension of the validity range of one-sided asymptotic formulae based on the procedure of small or large parameter 'elimination' (the method of asymptotic extrapolation). The material is presented on an example of convective mass and heat transfer and hydrodynamics problems. A comparison of the results obtained with a variety of specific cases, for which exact and approximate formulae are available shows good accuracy and wide possibilities of the methods suggested.

1. INTRODUCTION

ONE OF THE basic analytical methods used to investigate complex boundary-value problems is the method of perturbations in a small or large characteristic non-dimensional problem parameter admitting different specific modifications: the method of stretched coordinates, the method of matched asymptotic expansions, the method of two-scale expansions, etc. [1–3]. The solution, obtained by this method, is generally represented by several first terms of the asymptotic series and is applicable only in a sufficiently narrow range of large or small parameters (the resulting asymptotic expansions very often diverge or extremely slowly converge). This makes it impossible to assess the behaviour of the solution at intermediate (finite) values of the parameter and imposes substantial restrictions on the use of asymptotic formulae in engineering practice.

There were attempts made to improve the convergence of asymptotic expansions by the Shenks and Euler transformations, approximations by rational fractions, by the choice of natural coordinates, etc. (see e.g. refs. [1–4]).

Unfortunately, all of the methods listed lack universality and are extremely laborious; furthermore, their use requires sufficient terms of the asymptotic series (usually a minimum of three or four), which, as a rule, are not known in advance. Therefore, in the overwhelming majority of cases the research workers attempted to interpolate the asymptotic solution to an intermediate region by resorting to additional information (provided, e.g. by numerical solution of the problem considered).

Earlier, in the first part of this paper [5], two new analytical methods have been suggested for the investigation of mass and heat transfer problems: the method of asymptotic correction and the method of model equations and analogies. The first method allows, by using exact asymptotic expressions of the initial boundary-value problem, to effectively improve different kinds of approximate (engineering) formulae obtained earlier both on theoretical grounds and from experimental data. The second method provides the possibility to determine the basic integral characteristics of the solution of complex (non-linear) boundary-value problems by investigating much more simple model auxiliary equations with subsequent application of the analogy principle.

This part of the work presents some very simple and rather general techniques allowing one, on the basis of asymptotic formulae (without resorting to any additional information), to obtain approximate expressions which can be used directly in engineering practice. The material is presented on the examples of convective mass and heat transfer and hydrodynamics problems.

2. THE METHOD OF ASYMPTOTIC INTERPOLATION

Basic to the construction of further approximate (engineering) formulae are two-sided asymptotic expansions of mean Sherwood (Nusselt) numbers obtained for different kinds of situations of convective mass and heat exchange of particles, droplets and bubbles with a flow.

The most simple approximate determination of the mean Sherwood number over the whole range of Peclet numbers, $0 \leqslant Pe < \infty$, can be generally obtained by taking the sum of the main terms in the asymptotic expressions of the mean Sherwood number at small ($Pe \to 0$, $Sh/Sh_0 \to 1$) and large ($Pe \to \infty$, $Sh/Sh_0 \to BPe^m$) Peclet numbers; the formulae of the type $Sh/Sh_0 = 1 + BPe^m$, where Sh_0 is the mean Sherwood number corresponding to the case of a fixed particle at $Pe = 0$. However, as the comparison with the familiar

NOMENCLATURE

- A second coefficient of mean Sherwood number expansion in small Peclet numbers in formula (1)
- a radius of a spherically shaped droplet or particle; radius of a plane disk; characteristic particle dimension
- B coefficient determining the higher term of mean Sherwood number expansion at large Peclet numbers, formula (3)
- C dimensionless concentration in the problem with volumetric chemical reaction, C_*/C_s
- C_* concentration
- C_s concentration on particle surface
- C_∞ unperturbed concentration at infinity
- C_x resistance coefficient
- c dimensionless concentration, $\frac{C_\infty - C_*}{C_\infty}$
- c_p specific heat of liquid
- D diffusion coefficient
- F_s function determining surface reaction kinetics, equation (58)
- $\mathbf{f}$ dimensionless total force of body resistance
- $f_\parallel, f_\perp$ dimensionless resistance forces of the body of revolution in the case of its parallel or perpendicular position in translational flow
- G shear factor
- G_{ij} shear tensor components in Cartesian coordinate system, X_1, X_2, X_3
- $\mathbf{i}$ single directing liquid velocity vector at infinity
- K_s surface reaction rate constant
- K_v volumetric reaction rate constant
- k_s dimensionless surface reaction rate constant, aK_sD^{-1}
- k_v dimensionless volumetric reaction rate constant, $a^2K_vD^{-1}$
- m exponent of the main term of mean Sherwood number expansion for large Peclet numbers, formula (3)
- Nu mean Nusselt number (based on particle radius)
- $\mathbf{n}$ unit vector located in the plane of the body of revolution
- n exponent of the second-order term of mean Sherwood number expansion in small Peclet numbers, formula (1)
- Pe diffusional Peclet number, aUD^{-1}
- Pe thermal Peclet number, $aU\chi^{-1}$
- p complex parameter of Laplace–Carlson transformation
- q parameter, $k_s(k_s+1)^{-1}$
- Re Reynolds number $aU_\infty\nu^{-1}$
- r_* radial coordinate, $r = r_*/a$
- r, θ, φ spherical coordinate system fixed with respect to the particle
- $r - r_s(\theta, \varphi) = 0$ particle surface equation
- Sh mean Sherwood number (based on particle radius)
- Sh_0 mean Sherwood number corresponding to the case of a fixed particle at $Pe = 0$
- Sh_∂ mean Sherwood number corresponding to purely diffusional mode of reaction on particle surface
- T_* temperature in flow
- T_s temperature on particle surface
- T_∞ unperturbed temperature at infinity
- T dimensionless temperature, $\frac{T_\infty - T_*}{T_\infty - T_s}$
- t_* time
- t dimensionless time, $a^{-2}\chi t_*$
- U characteristic flow velocity
- U_∞ unperturbed flow velocity at infinity
- $\mathbf{v}$ dimensionless liquid velocity vector
- X_1, X_2, X_3 rectangular Cartesian coordinate system fixed with respect to the particle.

Greek symbols

- α numerical coefficient in the mean Sherwood number expansion at small Peclet numbers in the case of a shear flow around a particle, formula (59)
- β droplet to surrounding liquid viscosity ratio ($\beta = 0$ corresponds to a gas bubble, $\beta = \infty$, to a solid particle)
- $\varepsilon(Pe)$ small parameter, formula (42)
- $\varepsilon_n(Pe)$ terms of asymptotic sequencies in expansions (18) and (46)
- λ_* thermal conductivity of liquid, $\lambda_*(T_*)$
- $\lambda(T) = \lambda_*(T_*)/\lambda_*(T_\infty)$ dimensionless thermal conductivity
- ν kinematic viscosity of liquid
- ρ liquid density
- $\boldsymbol{\tau}$ directing vector of the axis of body of revolution
- χ thermal diffusivity of liquid
- ω angle between the axis of the body of revolution and free stream direction.

numerical results shows this technique of interpolation formulae construction usually leads to substantial errors and is of little value for practical application. This is because the parameter m, corresponding to the asymptotic expression for Sh at infinity, takes on the value $0 < m < 1$, thus leading to an infinite derivative at the zero for the above approximate relationship, i.e. $(dSh/dPe)_{Pe=0} = \infty$. In contrast, the exact value, corresponding to the derivative of the actual value of Sh, is usually limited. It is this very difference which is mainly responsible for the fact that the two-term approximate formulae, consisting of the sum of the main terms of asymptotic expressions for the mean Sherwood number, yield highly overestimated values at $Pe = 0.1–10$.

Here, the interpolation formulae will be constructed by another, much more exact, method making the best of the available asymptotic information about the mean Sherwood number. The idea behind the method consists in the following. Let the approximation for the asymptote to the mean Sherwood number at small Peclet numbers be determined, accurate to $o(Pe^n)$, by the expression

$$Sh/Sh_0 = 1 + APe^n \quad (Pe \to 0), \tag{1}$$

where the parameters A and n depend on the shape and the kind (solid, liquid or a bubble) of a particle and the type of a flow.

It is natural to seek the unknown approximate dependence of the mean Sherwood number on the Peclet number in the form of the following formula

$$Sh/Sh_0 = 1 + APe^n\Phi(Pe), \quad \Phi(0) = 1 \tag{2}$$

which retains the same accuracy order for $Pe \to 0$ as that of formula (1).

The function $\Phi(Pe)$ is assigned *a priori* on different grounds and, as a rule, is dependent on several parameters which are then selected so that approximate formula (2) would give, with a specified accuracy, a correct asymptotic result for the other limiting case when $P \to \infty$. In a particular case, when only the higher term in the mean Sherwood number asymptotic expression at large Peclet numbers is known

$$Sh/Sh_0 = BPe^m \quad (Pe \to \infty) \tag{3}$$

for such a function Φ one can take $\Phi = (1 + EPe^l)^q$, which at $q = -1$ leads, as a result of the limiting transition in equation (2) for $Pe \to \infty$ and subsequent comparison between equations (2) and (3), to the following most simple interpolation formula for the mean Sherwood number ($E = A/B$, $l = n-m$):

$$\frac{Sh}{Sh_0} = 1 + \frac{APe^n}{1+(A/B)Pe^{n-m}}. \tag{4}$$

This formula can already be used for practical purposes in the entire range of Peclet numbers $0 \leqslant Pe < \infty$ provided that the following conditions are fulfilled

$$A > 0, \quad B > 0, \quad n > m \geqslant 0, \tag{5}$$

the first two of which follow from the obvious fact that the mean Sherwood number increases monotonously with Pe ($dSh/dPe > 0$); the latter condition means that the rate of growth of the mean Sherwood number should decrease with an increasing Peclet number ($d^2Sh/dPe^2 < 0$).

By structure, approximate equation (4) ensures a correct asymptotic result in the limiting cases of small, equation (1), and large, equation (3), Peclet numbers. The comparison of relations (1) and (4) also shows that relation (4) gives an exact value of the derivative $dSh/dPe \neq \infty$ at zero for $Pe = 0$. The latter circumstance should lead to a noticeable improvement of the accuracy of interpolation formula (4) (as compared with the binomial $Sh/Sh_0 = 1 + BPe^m$) in the region of moderate Peclet numbers, $0.1 \leqslant Pe \leqslant 10$.

3. SPECIFIC EXAMPLES OF APPROXIMATE FORMULAE CONSTRUCTION BASED ON THE ASYMPTOTIC INTERPOLATION PROCEDURE

Now, the use of interpolation formula (4) will be illustrated for some specific situations of practical interest. Incidentally, when possible, a comparison will be made between the results obtained and available exact or approximate expressions, thus assessing the accuracy and the range of validity of interpolation formula (4). For simplicity the consideration will be restricted, for the time being, to the case of diffusively reacting particles, droplets and spherical bubbles, which corresponds to the value $Sh_0 = 1$ in equations (1)–(4) (here and hereafter when writing the dimensionless quantities Pe, Re, Sh, Nu, etc. the sphere radius a is taken to be the characteristic length scale).

3.1 *A moderately viscous bubble and droplet in a translational flow*

For a spherical bubble in a translational Stokes flow, the first two terms of the asymptotic expansion of the mean Sherwood number at small Peclet numbers are determined from formula (1) at $A = 1/2$, $n = 1$ [6]. In the other limiting case, when $Pe \to \infty$, the results of ref. [7] yield

$$Sh = (2Pe/3\pi)^{1/2} = 0.461\, Pe^{1/2},$$

which, with equation (4) taken into account, leads to the following interpolation formula

$$Sh = 1 + \frac{0.5Pe}{1+1.1Pe^{1/2}}, \quad Pe = \frac{aU_\infty}{D}, \tag{6}$$

where U_∞ is the unperturbed flow velocity at infinity, D the diffusion coefficient; here and hereafter all numerical coefficients in the final expression of the type of formula (6) will be rounded off to two significant figures.

The adequacy of approximate expression (6) at intermediate Peclet numbers was checked against numerical solution [8, 9] of the corresponding problem on mass transfer of a spherical bubble (Fig. 1). It is seen that the maximum error of equation (6) is 10–12%. It should be noted that the error of equation (6) in whose denominator the first term is discarded [and which, at $Pe \to 0$, gives only one correct term of expansion (1) and corresponds to the sum of the main terms in the asymptotic expressions of the mean Sherwood number at small and large Peclet numbers] will be more appreciable, i.e. about 18%.

Expression (4) at $A = 1/2$, $n = 1$, $B = 0.461\ (\beta + 1)^{-1/2}$, $m = 1/2$ [β is the ratio between the dynamic

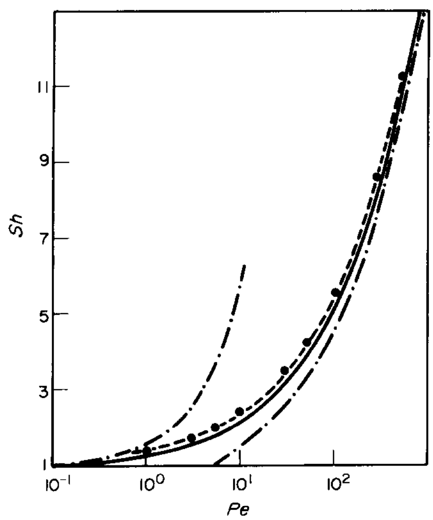

FIG. 1. The mean Sherwood number vs the Peclet number for the case of a translational Stokes flow around a spherical bubble: ——, calculation by formula (6); – – –, numerical calculations with the use of the finite-difference method [8]; ...., interpolation formula [9]; –.–.–., asymptotic expressions at small and large Peclet numbers.

viscosities of the droplet and surrounding liquid; $\beta = 0$ corresponds to a bubble, equation (6)] can be used to calculate a moderately viscous droplet, $0 \leqslant \beta \leqslant 1$; in this case the error of the approximate formula increases gradually with the parameter β, reaching 12–14% at $\beta = 1$.

3.2. *A solid sphere in a translational flow*

In the case of a solid sphere ($\beta = \infty$) at small Peclet numbers, formula (1) is valid at $A = 1/2, n = 1$. At large Peclet numbers, the main term in the asymptotic expression of the mean Sherwood number is determined as $Sh = 0.624\ Pe^{1/3}$ ($B = 0.624$, $m = 1/3$) [7], and this, with equation (4) taken into account, leads to the following approximate formula

$$Sh = 1+\frac{0.5Pe}{1+0.8Pe^{2/3}}, \quad Pe = \frac{aU_\infty}{D} \quad (\beta = \infty). \quad (7)$$

The comparison with the results of numerical solution for a corresponding problem on solid sphere mass transfer [10, 11] shows (see Fig. 2) that the maximum error of formula (7) over the entire range of Peclet numbers does not exceed 11–13% (the error of the respective approximate formula, which is given by the sum of the higher terms of asymptotic expressions at small and large Peclet numbers, is above 35% in this case).

3.3. *A solid sphere in a simple shear flow*

Consider the mass transfer of a diffusively reacting solid spherical particle freely suspended in a linear shear flow whose velocity field far from the particle has the following form in the Cartesian coordinate system X_1, X_2, X_3

$$r_* \to \infty, \quad \mathbf{V} = \{GX_2, 0, 0\}$$
$$(r_* = [X_1^2+X_2^2+X_3^2]^{1/2}), \quad (8)$$

where G is the shear factor.

The asymptotic expression for the mean Sherwood number, corresponding to the Stokes simple linear shear flow (at $Pe = 0$) (8) around a spherical particle at small Peclet numbers is given by formula (1) at $A = 0.257$, $n = 1/2$ [12]; at large Peclet numbers the following limiting property is valid: $\lim_{Pe\to\infty} Sh = 4.5$ ($B = 4.5$, $m = 0$) [13]. The boundedness of the mean Sherwood number at $Pe \to \infty$ is due to the presence, in this case, of the flow region with completely closed streamlines fully surrounding the particle and substantially decreasing the mass transfer rate. As before, the approximate relation for Sh is sought using expression (4) which, with the earlier asymptotic expressions taken into account, makes it possible to obtain the following approximate formula for the mean Sherwood number

$$Sh = 1+\frac{0.26Pe^{1/2}}{1+0.057Pe^{1/2}}. \quad (9)$$

3.4. *A solid sphere and a bubble in an arbitrary, purely deformational linear shear flow*

In the case of an arbitrary, purely deformational linear shear flow, the liquid velocity field far from the particle is governed by the expression

$$r_* \to \infty, \quad V_i = G_{ij}X_j$$
$$(G_{ij} = G_{ji}, G_{11}+G_{22}+G_{33} = 0), \quad (10)$$

where G_{ij} are the shear tensor components in the Cartesian coordinate system X_1, X_2, X_3; the diagonal

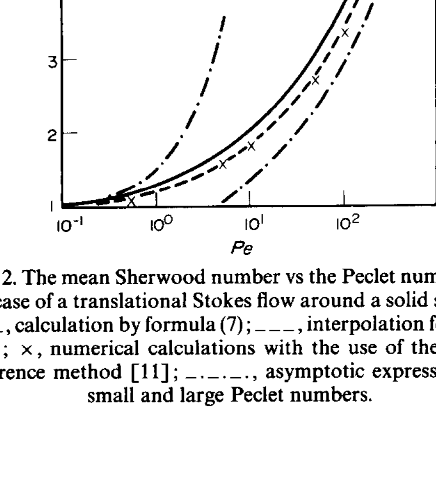

FIG. 2. The mean Sherwood number vs the Peclet number for the case of a translational Stokes flow around a solid sphere: ——, calculation by formula (7); – – –, interpolation formula [10]; ×, numerical calculations with the use of the finite-difference method [11]; –.–.–., asymptotic expressions at small and large Peclet numbers.

elements being equal to zero results from the liquid incompressibility condition div $\mathbf{V} = 0$; here and hereafter the summation is performed over the repeated indices $i,j = 1,2,3$. That the approximate formula for the mean Sherwood number could be constructed, use should be made of its asymptotic expression (1) at small Peclet numbers, where $A = 0.36$, $n = 1/2$ [14] and also of the asymptotic result $Sh \to 0.9Pe^{1/3}$ $(Pe \to \infty)$ [14] obtained in the diffusional boundary-layer approximation. With expression (4) ($B = 0.9$, $m = 1/3$) taken into account, it is possible to obtain the following formula

$$Sh = 1+\frac{0.36Pe^{1/2}}{1+0.4Pe^{1/6}}, \quad Pe = \frac{a^2(G_{ij}G_{ij})^{1/2}}{D}, \quad (\beta = \infty). \tag{11}$$

Now, an interpolation formula will be derived for the mean Sherwood number in the case of an arbitrary, purely deformational shear flow, equation (10), around a spherical bubble. The further analysis will require a somewhat greater information than is generally used. At small Peclet numbers [14, 15]

$$Sh = 1+0.36Pe^{1/2}+0.13Pe+O(Pe^{3/2}), \quad (Pe \to 0). \tag{12}$$

By extending the idea [14] on the characteristic velocity of a purely deformational linear shear flow around solid particles to bubbles and droplets with allowance for the respective results for an axisymmetric shear flow around a droplet [16], it is possible to obtain the following formula

$$Sh = 0.62(\beta+1)^{-1/2}Pe^{1/2},$$
$$Pe = \frac{a^2(G_{ij}G_{ij})^{1/2}}{D}, \quad (Pe \to \infty), \tag{13}$$

which allows, at large Peclet numbers, an approximate calculation of the mean Sherwood number per diffusively reacting moderately viscous spherical droplet $[0 \leqslant \beta < O(Pe^{1/3})]$ in an arbitrary, purely deformational linear shear Stokes flow. The case of a bubble corresponds to $\beta = 0$ in equation (13).

It is seen from equation (13) that the asymptotic expression for the mean Sherwood number at large Peclet numbers has the order of $Pe^{1/2}$ and leads to the value $m = n$ in equation (4), i.e. here the constant E cannot be already selected so that it would be possible to simultaneously obtain the two-term asymptotic expression (12) (accurate to the terms of order $Pe^{1/2}$ inclusive) at small Peclet numbers and the one-term asymptotic expression (13) at large Peclet numbers. Therefore, for the present case of a spherical bubble, the three-term expansion (12) at small Peclet numbers with the factor $(1+EPe^l)^{-1}$ in the third term will be taken as a basis for the construction of the respective approximate formula. Complying with the requirement that the main term of the asymptotic expression for the approximate formula be in accord with the result of equation (13) when $Pe \to \infty$, the constants E and l can be determined thus leading to the following interpolation formula for the mean Sherwood number

$$Sh = 1+0.36Pe^{1/2}+\frac{0.13Pe}{1+0.5Pe^{1/2}},$$
$$Pe = \frac{a^2(G_{ij}G_{ij})^{1/2}}{D}, \quad (\beta = 0). \tag{14}$$

Expressions (9), (11), (14) allow an approximate calculation of the mean Sherwood number per diffusively reacting solid spherical particle and bubble in different types of linear shear flow in the entire range of Peclet numbers, $0 \leqslant Pe < \infty$. Unfortunately, there are not any numerical results at present which would make it possible to assess the accuracy of the proposed approximate formulae. By analogy with the translational flow, it should be expected that the maximum error of these approximate expressions would amount to 10–12% (it should be noted that in the case of a simple shear flow it is impossible to construct an approximate formula for the Sherwood number by simply summing up the main terms of asymptotic expressions for $Pe \to 0$ and $Pe \to \infty$).

It is also not difficult to construct analogous approximate formulae for the mean Sherwood number in the case of nonspherical particles. Thus, for a plane disk, whose axis is directed along the flow, it is possible to use the asymptotic results [6, 17] which are respectively valid at small and large Peclet numbers. Omitting the intermediate calculations carried out with formula (4) taken into account, the following equation can be obtained in this case for the mean Sherwood number

$$\frac{Sh}{Sh_0} = 1+\frac{0.5\,Sh_0Pe}{1+0.65\,Sh_0Pe^{3/4}}, \quad Sh_0 = \frac{2}{\pi} = 0.64. \tag{15}$$

The dimensionless total diffusional flux per surface is converted by the formula $I = 4\pi Sh$; when writing down dimensionless expression (15), the radius of the disk was taken to be the characteristic length scale.

3.5. *The resistance coefficient of a spherical bubble*

Now the above method of the construction on interpolation formulae will be used to obtain an approximate relation between the resistance coefficient of a spherical bubble C_x and the Reynolds number, $Re = aU_\infty/\nu$ (ν is the kinematic viscosity coefficient). The limiting relations for the resistance coefficient which have been obtained at small and large Reynolds numbers in refs. [18] and [19] are, respectively,

$$Re \to 0,\ C_x = \frac{8}{Re}+1; \quad Re \to \infty,\ C_x = \frac{24}{Re}. \tag{16}$$

By isolating the complex $\gamma = \frac{1}{8}ReC_x$ of asymptotic expression (16), it is possible to write the above in a more simple form

$$Re \to 0,\ \gamma = 1+\tfrac{1}{8}Re; \quad Re \to \infty,\ \gamma = 3,$$

thus allowing a direct use of expression (4) in which it should be assumed that $Sh/Sh_0 = \gamma$, $A = 1/8$, $B = 3$, $m = 0$. The above procedure allows the following

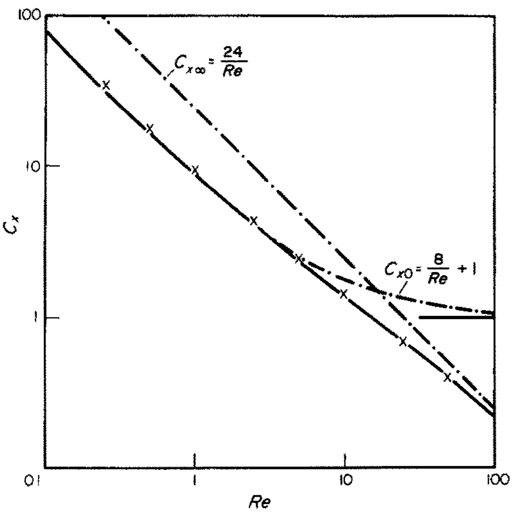

FIG. 3. The resistance coefficient of a spherical bubble vs the Reynolds number: ——, calculation by formula (17); ×, numerical calculation with the use of the finite-difference method [20]; —·—·—·, asymptotic expressions (16).

interpolation formula to be obtained

$$C_x = \frac{8}{Re} + \frac{1}{1+\frac{1}{16}Re}, \tag{17}$$

which makes it possible to calculate the resistance coefficient of a spherical bubble in the entire range of Reynolds numbers.

Figure 3 compares formula (17) with the results [20] of numerical solution of the respective problem of flow around a bubble. It is seen that the maximum error of approximate formula (17) is less than 4.5%.

4. THE METHOD OF ASYMPTOTIC EXTRAPOLATION

In those cases when only the one-sided (e.g. when $Pe \to 0$) asymptotic expression of the sought-after quantity is known, it often becomes possible to use the method of asymptotic extrapolation which enables the region of applicability of asymptotic formulae to be substantially extended. As before, the material will be presented with the problems of convective mass and heat transfer used as examples.

It will be assumed that in addition to the small or large parameter Pe, in which the perturbation is made, there is one more characteristic (finite) parameter k in the problem. It is the presence of this free unperturbed parameter, k, which allows one to 'eliminate' the small parameter Pe from the initial asymptotic formula by different means. This, ultimately, gives the possibility of obtaining different kinds of asymptotically approximate formulae which then could be used for practical computations in a much wider range of Peclet numbers. First, the method will be described for the simplest situation (see also Section 6).

Let for the mean Sherwood number the method of perturbations in the small (large) parameter Pe give the $(N+1)$-term asymptotic expansion of the form $(N \geqslant 1)$

$$Sh(k, Pe) = \sum_{n=0}^{N} \varepsilon_n(Pe)A_n(k)\left(\lim_{Pe\to 0} \frac{\varepsilon_{n+1}}{\varepsilon_n} = 0\right). \tag{18}$$

The assumption of $k = 0$ in equation (18) ($k = 0$ can be replaced also by $k = \infty$) will give

$$Sh(0, Pe) = \sum_{n=0}^{N} \varepsilon_n(Pe)A_n(0). \tag{19}$$

Now, the elimination of the small parameter Pe from equation (19) through the use of $Sh(0, Pe)$ and substitution of the latter into equation (18) will yield

$$Sh(k, Pe) = \Phi(k, Sh(0, Pe)). \tag{20}$$

If now it can be 'forgotten' that the coefficient $Sh(0, Pe)$ in expression (20) is prescribed by equation (19), it will be determined directly from the solution of the initial problem at $k = 0$, $Pe =$ const. [the solution of the problem in the specific case of $k = 0$ can be performed in a much simpler way than at $k \neq 0$; note also that in a number of cases the quantity $Sh(0, Pe)$ can also be determined directly from an experiment], then equation (20) can be used for the approximate determination of the mean Sherwood number in a much more wider range of Peclet numbers than the initial asymptotic formula (18). This is due to the fact that, besides a correct description of the solution for $Pe \to 0$, formula (20) gives also an exact result for the other limiting case at $k = 0$ [this latter property is not characteristic of the initial asymptotic formula (18)]. Therefore, approximate expression (20) is basically an extrapolation of asymptotic formula (18) to the region of small values of the parameter k (where Pe is any value), which, as a rule, in turn substantially extends its validity range.

In a specific case of two-term expansion (18) at $N = 1$ and $\varepsilon_0(Pe) = 1$, expression (20) acquires the following simple form

$$Sh(k, Pe) = A_0(k) + \frac{A_1(k)}{A_1(0)}[Sh(0, Pe) - A_0(0)] \tag{21}$$

$$(0 < |A_1(0)| < \infty).$$

The method of asymptotic extrapolation admits different complexities and modifications. Thus, the situation may be such that it is impossible to obtain the asymptotic solution of the initial problem for $Pe \to 0$ or $Pe \to \infty$ ($k =$ const.). However, if in this case an asymptotic solution of the problem can be found in the limit [$\Psi = \Psi(Pe)$ is a certain function of the parameter Pe]

$$Pe \to 0, \quad k = \Psi(Pe)\zeta, \quad \zeta = 0(1), \tag{22}$$

then first it is necessary in the respective expansion for Sh

$$Sh = \sum_{n=0}^{N} \delta_n(Pe)B_n(\zeta), \quad \lim_{Pe\to 0} \frac{\delta_{n+1}}{\delta_n} = 0 \tag{23}$$

to express ζ in terms of k ($\zeta = k/\Psi$) and then, to the resulting expression to apply the procedure of asymptotic correction [5] with respect to both parameters Pe and k simultaneously (see examples in Section 5).

It should be noted that the method of asymptotic extrapolation is an important particular case of a more general method of asymptotic correction [5]. Namely, these methods coincide when a truncated series of the form of equations (18), or (22), (23) is selected as an initial approximate formula which needs the improvement by the method of asymptotic correction.

The above will be illustrated on specific examples.

5. SPECIFIC EXAMPLES OF THE CONSTRUCTION OF APPROXIMATE FORMULAE BASED ON THE ASYMPTOTIC EXTRAPOLATION PROCEDURE

Consider convective diffusion to a solid sphere in a laminar translational shear flow of a viscous incompressible liquid. A first-order heterogeneous chemical reaction is assumed to take place on the particle surface. In dimensionless variables, the corresponding boundary-value problem has the form

$$Pe(\mathbf{v}\cdot\nabla)c = \Delta c$$

$$r = 1,\ \frac{\partial c}{\partial r} = k_s(c-1);\quad r \to \infty,\ c \to 0 \tag{24}$$

$$c = \frac{C_\infty - C_*}{C_\infty},\quad k_s = \frac{aK_s}{D},\quad Pe = \frac{aU_\infty}{D}.$$

Here, C_* and C_∞ are the concentrations in the flow and at infinity, respectively and K_s is the surface chemical reaction rate constant.

The asymptotic analysis of problem (24) has been carried out at small Peclet numbers $Pe \ll 1$, and has resulted in the following expression for the mean Sherwood number

$$Sh(k_s, Pe) = q + \frac{1}{2}q^2 Pe,\ q = \frac{k_s}{k_s+1} \tag{25}$$

$$\left(Sh = -\frac{1}{2}\int_{-1}^{1}\left(\frac{\partial c}{\partial r}\right)_{r=1} \mathrm{d}\mu,\ \mu = \cos\theta\right).$$

At $k_s = 0$, the solution of problem (24) is trivial $c = 0$ and corresponds to the zero value of the auxiliary Sherwood number $Sh(0, Pe) = 0$, which also follows from formula (25). Therefore it is necessary to replace $k_s = 0$ in formula (25) by $k_s = \infty$, which corresponds to $Sh(\infty, Pe) = 1 + \frac{1}{2}Pe$. Representing here Pe in terms of $Sh(\infty, Pe)$ and substituting the latter into formula (25) yield the following expression, which is analogous to formula (21)

$$Sh(k_s, Pe) = q + q^2[Sh(\infty, Pe) - 1]. \tag{26}$$

The comparison with the results of the corresponding numerical solution [11] of problem (24) {the flow velocity field $\mathbf{v}$ in ref. [11] was also determined by numerical solution of the corresponding hydrodynamic problem of viscous incompressible liquid flow around a sphere; for comparison, the value of $Sh(\infty, Pe)$ in expression (26) was taken from ref. [11]} shows that the accuracy of approximate formula (26) by more than an order exceeds the accuracy of initial asymptotic formula (25). In particular, expression (26) may well be used at moderate Peclet numbers within the range $0 \leqslant Pe \leqslant 10$ [the maximum error of formula (26) in this case is about 14%], while the region of applicability of initial asymptotic formula (25) is limited by small Peclet numbers, $0 \leqslant Pe \leqslant 0.5$.

Consider now the steady-state convective mass transfer between a spherical droplet or a solid particle and a translational Stokes flow with first-order volumetrical chemical reaction occurring in the liquid. In dimensionless variables, the respective boundary-value problem has the form

$$k_v C + Pe(\mathbf{v}\cdot\nabla)C = \Delta C \tag{27}$$

$$r = r_s(\theta, \varphi),\ C = 1;\quad r \to \infty,\ C \to 0 \tag{28}$$

$$C = C_*/C_s,\quad k_v = a^2 K_v/D.$$

Here $r = r_s(\theta, \varphi)$ is the equation for the particle surface, C_s the concentration on the particle surface, K_v the volumetric chemical reaction rate constant; the liquid velocity field $\mathbf{v}$ in equation (27) should be determined by solving a respective hydrodynamic problem of the flow around a particle.

It does not appear possible to investigate the problem (27), (28) by the asymptotic methods for $Pe \to 0$, $k_v = \text{const.}$ (or when $Pe \to \infty$, $k_v = \text{const.}$). Nevertheless, it turns out that in the case of a translational viscous incompressible liquid flow around a droplet or an arbitrarily shaped solid particle, it is possible to carry out the asymptotic analysis of problem (27), (28) in the limit when

$$Pe \to 0,\quad k_v = Pe^2\zeta,\quad \zeta = 0(1)\quad (\zeta = \text{const.}). \tag{29}$$

This case corresponds to the smallness of both parameters Pe and k_v simultaneously [see formula (22) at $\Psi(Pe) = Pe^2$].

It follows from the results of work [23] that the two-term asymptotic expansion of the mean Sherwood number determined from the solution of problem (27), (28) under condition (29) is of the form

$$Sh = Sh_0 + \tfrac{1}{2}PeSh_0^2(1+4\zeta)^{1/2}, \tag{30}$$

where Sh_0 is the mean Sherwood number corresponding to the mass transfer between a particle and a stationary medium in the absence of a volumetric chemical reaction; the error of formula (30) is of the order of $Pe^2 \ln Pe$.

Now, introduce the Peclet number under the radical and express the parameter ζ in terms of k_v by employing the relationship between these quantities (29). This results in the following expression for the mean Sherwood number

$$Sh(k_v, Pe) = Sh_0 + \tfrac{1}{2}(Pe^2 + 4k_v)^{1/2}Sh_0^2 \quad (Sh_0 = Sh(0,0)). \tag{31}$$

In the absence of the volumetric chemical reaction $k_v = 0$ ($\zeta = 0$), formulae (30) and (31) pass over, accurate to o(Pe), to the results of ref. [6].

Following the procedure, described earlier in Section 4, and assuming successively in formula (31) that $k_v = 0$ and $Pe = 0$ yield the relations

$$Sh(0, Pe) = Sh_0 + \tfrac{1}{2} Pe Sh_0^2,$$
$$Sh(k_v, 0) = Sh_0 + k_v^{1/2} Sh_0^2. \qquad (32)$$

From this, expressing the parameters k_v and Pe and substituting them into formula (31) results in the following approximate formula

$$Sh(k_v, Pe) = Sh_0 + \{[Sh(0, Pe) - Sh_0]^2 + [Sh(k_v, 0) - Sh_0]^2\}^{1/2}. \qquad (33)$$

Further, as always, it will be assumed that the parameters $Sh(0, Pe)$ and $Sh(k_v, 0)$ from formula (33) can now be determined by solving the corresponding auxiliary boundary-value problems (27) and (28) at $k_v = 0$ and $Pe = 0$, respectively [i.e. hereafter relations (32) are 'forgotten'].

By structure, formula (33) gives an exact result in the limiting cases $k_v = 0$ (Pe = const.) and $Pe = 0$ (k_v = const.). Moreover, in view of the following properties of the auxiliary Sherwood numbers: $Sh(0, Pe) \to \infty$ when $Pe \to \infty$ (here only those cases are considered when there are specific flow stagnation points on the particle surface and the situation [12, 13], when the particle is surrounded by the flow region with completely closed streamlines, is not investigated) and $Sh(k_v, 0) \to \infty$ when $k_v \to \infty$, expression (33) yields the limiting equalities $Pe \to \infty$ (k_v = const.), $Sh(k_v, Pe) \to Sh(0, Pe)$ and $k_v \to \infty$ (Pe = const.), $Sh(k_v, Pe) \to Sh(k_v, 0)$, which, as can be easily verified, are also asymptotically correct for the initial boundary-value problem (27), (28). In other words, approximate formula (33) obtained from asymptotic expression (30), which was derived on the assumption, formula (29), of the smallness of both parameters k_v and Pe simultaneously, turns to be applicable for all possible limiting cases of large and small parameters k_v and Pe.

The specific case of a spherical droplet or solid particle corresponds to $Sh_0 = 1$ and $Sh(k_v, 0) = 1 + \sqrt{k_v}$ in expression (33), which leads to the following formula

$$Sh(k_v, Pe) = 1 + \{[Sh(0, Pe) - 1]^2 + k_v\}^{1/2}. \qquad (34)$$

When using this formula for $Sh(0, Pe)$, an exact value of the mean Sherwood number should be selected which corresponds to a purely diffusional regime of mass transfer between a droplet or a solid particle and a flow without chemical reaction ($k_v = 0$).

The comparison of approximate expression (34) with the results of ref. [24] obtained in the diffusional boundary layer approximation ($Pe \gg 1$) for a Stokes flow around a spherical drop shows that in this case the maximum error of formula (33) for the whole range of the dimensionless volumetric chemical reaction rate constant, $0 \leqslant k_v < \infty$, is only 7%. Here, it will be emphasized once again that the validity range of the initial asymptotic formula (31) was limited by small values of both parameters k_v and Pe simultaneously: $0 \leqslant k_v \approx Pe^2 < Pe \ll 1$.

It should be noted that at large Peclet numbers expression (34) passes over, at $n = 1$, into formula (16), derived earlier in Part I of this paper by quite a different method [5]. Formula (34) has the same advantages over formula (16) [5] that, besides large Peclet numbers, it can also be used at small and intermediate Peclet numbers.

Consider now the unsteady-state convective heat and mass transfer to a diffusively reacting solid or liquid spherical particle in a steady translational (viscous) incompressible liquid flow. It is assumed that at $t < 0$ the temperature in the flow and on the particle surface is constant and equal to T_∞, and that at $t = 0$ the particle temperature has changed suddenly ('stepwise') to a constant value, T_s, which then is maintained unchanged during the whole process. In dimensionless variables, the corresponding unsteady-state boundary-value problem is formulated as

$$\frac{\partial T}{\partial t} + Pe(\mathbf{v} \cdot \nabla) T = \Delta T \qquad (35)$$

$$t = 0,\ T = 0; \quad r = 1,\ T = 1; \quad r \to \infty,\ T \to 0 \qquad (36)$$

$$T = \frac{T_\infty - T_*}{T_\infty - T_s}, \quad t = \frac{\chi t_*}{a^2}, \quad Pe = \frac{a U_\infty}{\chi},$$

where T_* is the temperature in the flow, t_* the time, χ the thermal diffusivity of the liquid.

Apply the Laplace–Carlson transformation to the equation, initial and boundary conditions (35), (36)

$$\bar{T} = p \int_0^\infty e^{-pt} T \, dt.$$

This will give

$$p\bar{T} + Pe(\mathbf{v} \cdot \nabla)\bar{T} = \Delta \bar{T}; \quad r = 1,\ \bar{T} = 1; \quad r \to \infty,\ \bar{T} \to 0. \qquad (37)$$

Problem (37) coincides, accurate to $p \to k_v$, $\bar{T} \to c$, with equations (27) and (28). Therefore, in the limit

$$Pe \to 0, \quad p = Pe^2 \zeta, \quad |\zeta| = 0(1), \qquad (38)$$

taking into account the fact that for a spherical particle $Nu_0 = 1$, it is possible to obtain a two-term asymptotic expansion, similar to equation (31), for the mean Nusselt number transfer [23]

$$\overline{Nu} = 1 + \tfrac{1}{2}(Pe^2 + 4p)^{1/2}. \qquad (39)$$

Complex limiting transition (38) corresponds to small Peclet numbers and large characteristic times of the process, $t \sim Pe^{-2}$.

Now assuming successively in formula (39) that $Pe = 0$ and $p = 0$ and proceeding further in the same way as for the problem of mass transfer with volumetric chemical reaction, it is possible, taking into account

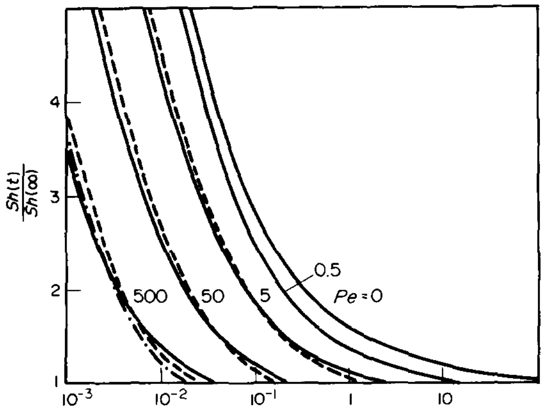

FIG. 4. The normalized mean Sherwood number vs time: ——, calculation by formula (41); – – –, numerical calcualtions with the use of the finite-difference method [11].

the equalities $\overline{Nu}(p,0) = 1+\sqrt{p}$ and $\overline{Nu}(0,Pe) = Nu(0,Pe)$, to obtain the following expression

$$\overline{Nu}(p,Pe) = 1+\{[Nu(0,Pe)-1]^2+p\}^{1/2}. \quad (40)$$

Then, on applying to this equation the Laplace–Carlson inversion formula, it is possible to find the following sought-after dependence of the mean Nusselt number on time

$$Nu = \frac{1}{\sqrt{(\pi t)}} \exp(-\xi^2 t)+\xi \operatorname{erf}(\xi\sqrt{t})+1, \quad (41)$$

$$Nu = Nu(t,Pe),\ \xi = Nu(\infty,Pe)-1.$$

This formula provides the correct result in all the limiting cases at large and small values of the parameter Pe and dimensionless time t.

The approximate dependence of the mean Nusselt number on time is of a rather general character and can be used to calculate the unsteady-state heat and mass transfer of solid particles, droplets and spherically shaped bubbles in flows of different geometry at any Peclet numbers. Figure 4 presents the comparison of approximate expression (41) (solid lines) with the results of work [11] obtained by numerical solution of problems (35), (36) by the network method for the case of a translational Stokes flow around a solid sphere at $Pe = 0.5, 5, 50, 500$ (dashed lines). The dashed–dotted line is plotted for $Pe = 500$ from the data of work [25] where the problem of unsteady-state transfer (35), (36) was investigated at large Peclet numbers in the diffusional boundary-layer approximation. It is seen that the maximum divergence between formula (41) and the results of work [11] at $Pe \leqslant 50$ is less than 10%.

6. SOME MODIFICATIONS OF THE METHOD OF ASYMPTOTIC EXTRAPOLATION

Now, some possible modifications of the method of asymptotic extrapolation, which are based on somewhat different grounds than those dealt with in Section 4, will be considered. For simplicity, the analysis will be restricted to the problems which depend only on two dimensionless parameters k and Pe.

Let the following two-term asymptotic expansion be obtained by the perturbation method in small (large) parameter Pe for the mean Sherwood number

$$Sh(k,Pe) = A(k)+\varepsilon(Pe)B(k)\left(\lim_{Pe\to 0} \varepsilon(Pe) = 0\right). \quad (42)$$

Assuming that the coefficients $A(k)$ and $B(k)$ are limited in the whole range of the parameter k, it will be adopted in formula (42) that $k = 0$ and $k = \infty$. This yields

$$Sh(0,Pe) = A(0)+\varepsilon(Pe)B(0) \quad (43)$$

$$Sh(\infty,Pe) = A(\infty)+\varepsilon(Pe)B(\infty).$$

Using these relations, it is not difficult to show that asymptotic formula (42) can be rewritten in the following equivalent form

$$Sh(k,Pe) = \varphi(k)Sh(0,Pe)+\psi(k)Sh(\infty,Pe)$$

$$\varphi(k) = \frac{A(k)B(\infty)-A(\infty)B(k)}{A(0)B(\infty)-A(\infty)B(0)},$$

$$\psi(k) = \frac{A(0)B(k)-A(k)B(0)}{A(0)B(\infty)-A(\infty)B(0)}. \quad (44)$$

It was assumed for equations (44) that the mean Sherwood number (42) can be presented in the form of a linear combination of the auxiliary Sherwood number, equations (43).

Expression (44) does not already contain the small parameter $\varepsilon(Pe)$ in its explicit form. If now it will be 'forgotten' that the coefficients $Sh(0,Pe)$ and $Sh(\infty,Pe)$ in formula (44) are specified by equalities (43) and they will be determined directly from the solution of the initial problem at $k = 0$, $Pe =$ const. and $k = \infty$, $Pe =$ const. [the solutions of auxiliary problems in the specific cases of $k = 0$ and $k = \infty$ can be much simpler than at $0 < k < \infty$; note also that in a number of cases the quantities $Sh(0,Pe)$ and $Sh(\infty,Pe)$ can be determined directly from experiment or calculated by numerical methods], then expression (44) can be used for the approximate determination of the mean Sherwood number in a much wider range of Peclet numbers. This is due to the fact that formula (44), in addition to the limiting case $Pe \to 0$, provides also a correct result for $k \to 0$ ($Pe =$ const. is any fixed number) and for $k \to \infty$ ($Pe =$ const.); these latter two properties are not typical of initial asymptotic formula (42).

There presentation of formula (44) is valid provided $A(0)B(\infty)-A(\infty)B(0) \neq 0$. Consider now a very important specific case of expansion (42) at $B(k) = \sigma A(k)$, $\sigma =$ const., for which it is already impossible to use formula (44). Assuming that $B = \sigma A+\delta$ in formula (44) and letting $\delta \to 0$, it is possible to obtain in this case

$$Sh(k, Pe) = \frac{A(k)-A(\infty)}{A(0)-A(\infty)} Sh(0, Pe) + \frac{A(0)-A(k)}{A(0)-A(\infty)} Sh(\infty, Pe) \quad (45)$$

$$(B(k) = \sigma A(k), \quad \sigma = \text{const.}).$$

It should be noted that formulae (44) and (45) remain valid also in a more general case, when for the mean Sherwood number at small Peclet numbers the following $(N+1)$-term expansion holds

$$Sh(k, Pe) = A(k) \sum_{n=0}^{N-1} \varepsilon_n(Pe) q_n + B(k)\varepsilon_N(Pe) \quad (46)$$

$$\lim_{Pe \to 0} (\varepsilon_{n+1}/\varepsilon_n) = 0, \quad q_n = \text{const.}, \quad N \geqslant 1.$$

The use of formulae (45) and (46) will be now illustrated on several specific examples, which are of interest in themselves.

For the further analysis, the following three-term asymptotic expansion [6] of the mean Sherwood number will be needed

$$Sh = Sh_0 + \tfrac{1}{2} Pe Sh_0^2 + \tfrac{1}{2} Pe^2 \ln Pe Sh_0^3 (\mathbf{f} \cdot \mathbf{i}) \quad (Pe \to 0), \quad (47)$$

corresponding to mass transfer of a diffusively reacting, arbitrarily shaped solid or liquid particle in a translational Stokes flow; here **f** is the dimensionless vector equal to the ratio between the resistance force of this particle to the Stokes resistance force of a volume-equivalent solid sphere of radius a, **i** is the unit directing liquid velocity vector at infinity.

6.1. *Mass transfer of a spherical arbitrarily viscous droplet*

In the case of a spherical droplet in a Stokes flow, the mean Sherwood number is determined by formula (47) where [6]

$$Sh_0 = 1, (\mathbf{f} \cdot \mathbf{i}) = \frac{2+3\beta}{3+3\beta}. \quad (48)$$

The substitution of relations (48) into expression (47) leads to three-term expansion (46) where $k = \beta, N = 2$, $A = 1$, $B = (2+3\beta)/(6+6\beta)$. Then, the use of formula (44) yields

$$Sh(\beta, Pe) = \frac{1}{\beta+1} Sh(0, Pe) + \frac{\beta}{\beta+1} Sh(\infty, Pe), \quad (49)$$

where $Sh(0, Pe)$ and $Sh(\infty, Pe)$ are the mean Sherwood numbers for a gas bubble ($\beta = 0$) and a solid sphere ($\beta = \infty$). If now the coefficients $Sh(0, Pe)$ and $Sh(\infty, Pe)$ in expression (49) would be determined from the solution of a complete diffusional problem without assuming the smallness of the Peclet number [in particular, the parameters $Sh(0, Pe)$ and $Sh(\infty, Pe)$ in expression (49) can be calculated from formulae (6) and (7)], then formula (49) can be used to approximately calculate the mean Sherwood number in a substantially wider range of Peclet numbers than does the initial asymptotic expansion (47), (48).

It should be noted that formula (49) was suggested earlier in refs. [11, 26] on a purely empirical basis for an approximate description of the mean Sherwood number dependence on the Peclet number within the range $0.5 \leqslant Pe \leqslant 500$. From the above method of expression (49) derivation and from the comparison [26] of this formula with the available numerical results it follows that this formula can be successfully used within the range $0 \leqslant Pe \leqslant 500$ (in this case its maximum error does not exceed 14%).

Formula (49) can presumably also be used for the calculation of the mean Sherwood number in the case of an arbitrary purely deformational linear shear flow around a spherical droplet; then expressions (14) and (11) could be used as the parameters $Sh(0, Pe)$ and $Sh(\infty, Pe)$.

6.2. *Heat and mass transfer to a body of revolution arbitrarily oriented in a translational flow*

It will now be shown in which way asymptotic formula (20) may yield another non-trivial information about the mass transfer of bodies of revolution arbitrarily orientated in a translational flow. It is assumed that the axis of the body of revolution makes up the angle ω with the flow velocity direction at infinity (Fig. 5). The directing liquid velocity vector **i** can be represented in the following way in terms of the directing vector of the body of revolution $\boldsymbol{\tau}$ and unit vector **n**, lying in the plane of the body of revolution

$$\mathbf{i} = \boldsymbol{\tau} \cos \omega + \mathbf{n} \sin \omega. \quad (50)$$

That the full hydrodynamic problem on the flow around a body of revolution can be solved, consider two auxiliary problems which correspond to two separate terms in equation (50) and which are determined by the following boundary conditions on the body surface and at infinity

$$r = r_s(\theta, \varphi), \mathbf{v}_{\parallel} = 0; \quad r \to \infty, \mathbf{v}_{\parallel} = \boldsymbol{\tau} \cos \omega \quad (51)$$

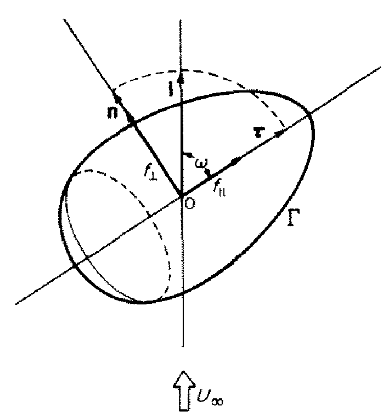

FIG. 5. A body of revolution in a translational flow. The case of arbitrary orientation.

$$r = r_s(\theta, \varphi),\ \mathbf{v}_\perp = 0; \quad r \to \infty,\ \mathbf{v}_\perp = \mathbf{n} \sin \omega. \quad (52)$$

Here, the motion and continuity equations are omitted. In the Stokes approximation, the motion equations are linear and the solution of the full problem will be governed by the superposition of the solution of auxiliary problems and boundary conditions (51) and (52). It follows from the results of work [27] that in the Stokes approximation the direction of the body resistance forces, corresponding to solution (51) and (52), coincides with the direction of the vectors $\boldsymbol{\tau}$ and $\mathbf{n}$. It is clear from what has been said that the dimensionless total body resistance force can be represented as

$$\mathbf{f} = f_\parallel \boldsymbol{\tau} \cos \omega + f_\perp \mathbf{n} \sin \omega, \quad (53)$$

where $f_\parallel$ and $f_\perp$ are the dimensionless resistance forces of the body of revolution in the case of its parallel ($\omega = 0$) and normal ($\omega = \pi/2$) position in a translational flow.

Expressions (50) and (53) allow one to calculate the scalar product

$$(\mathbf{f} \cdot \mathbf{i}) = f_\parallel \cos^2 \omega + f_\perp \sin^2 \omega. \quad (54)$$

The substitution of equation (54) into formula (47) leads to a three-term expansion (46) at $k = \omega$, $N = 2$, $A = 1$, $B = (\mathbf{f} \cdot \mathbf{i})$. The use of expression (44), in which ∞ should be replaced by $\pi/2$, gives the possibility to represent the mean Sherwood number as

$$Sh = Sh_\parallel \cos^2 \omega + Sh_\perp \sin^2 \omega, \quad (55)$$

where $Sh_\parallel$ and $Sh_\perp$ are the mean Sherwood numbers corresponding to parallel and perpendicular position of the body of revolution in a translational flow. Formula (55) gives the possibility to approximately determine the mean Sherwood number for any orientation of the body of revolution in a flow at any Peclet numbers. Since for a spherically shaped particle equality (55) is fulfilled identically at any Peclet numbers, it should be then expected that for the particles, whose shape is close to a spherical one, the approximate formula (55) will give good results not only at small, but also at intermediate and large Peclet numbers.

It follows from the results of ref. [28] that formulae (49) and (55) may also be used for the case of the first-order heterogeneous chemical reaction occurring on the surface of a reacting droplet or particle.

It should be noted that approximate formulae (49) and (55) were constructed on the basis of the dependence of the logarithmic term in asymptotic expansions (47), (48) and (47), (54) in small Peclet number on the second finite parameter of the problem (β and ω). This circumstance is of particular interest since, when the initial asymptotic expressions of type (47), (48), (54) are used directly, the logarithmic terms as such are not of practical importance until the next algebraic term in the expansion has been calculated [1]. This is due to two reasons: first, a change in the perturbation value transfers the logarithmic term constant into the consecutive algebraic term and, second, the logarithmic terms have a distinct nonmonotonous character (at $0 \leqslant Pe \leqslant 1$). In this sense, both previous examples of the construction of approximate formulae (49) and (55) represent a sort of 'rehabilitation' for the logarithmic terms in asymptotic expansions.

6.3. *Resistance force of an arbitrarily viscous droplet*

In ref. [18], for the resistance coefficient of an arbitrarily viscous spherical droplet in a translational flow at small Reynolds numbers, the following asymptotic expansion has been obtained

$$C_x(\beta, Re) = \frac{3\beta+2}{\beta+1}\left(\frac{4}{Re} + \frac{Re}{2} + \frac{1}{40} Re^2 \ln Re\right),$$

$$Re = \frac{aU_\infty}{\nu}, \quad (56)$$

which, apart from obvious new designation, corresponds to the case $A = B$ in two-term expression (42) [or three-term expression (46)]. Therefore, by taking use of formula (45), where it should be assumed that $Sh = C_x$, $k = \beta$, $Pe = Re$, $A = (3\beta+2)/(\beta+1)$, it is possible to rewrite expression (56) in the following equivalent form

$$C_x(\beta, Re) = \frac{1}{\beta+1} C_x(0, Re) + \frac{\beta}{\beta+1} C_x(\infty, Re), \quad (57)$$

where $C_x(0, Re)$ and $C_x(\infty, Re)$ are the resistance coefficients for a bubble and a solid sphere.

Formula (57) was suggested earlier [20] on a purely intuitive basis; it was also shown there that its error was less than 5% at $0 \leqslant Re \leqslant 50$. It should be noted that the solution of the auxiliary problems to determine the coefficients $C_x(0, Re)$ and $C_x(\infty, Re)$ turns to be much simpler than the solution of the initial problem at $0 < \beta < \infty$ (since in the limiting cases of $\beta = 0$ and $\beta = \infty$ only the external region of the flow is considered, while at $0 < \beta < \infty$, both the external and internal flow regions are considered simultaneously).

Formulae (44) and (45) corresponded to the case of linear extrapolation of the asymptotic results to the intermediate region. In some cases it also turns possible to effectively use a nonlinear extrapolation which should be illustrated on the following example.

6.4. *Mass transfer of a particle with arbitrary chemical reaction occurring on its surface*

Now, consider the mass transfer of a spherical particle in a translational and shear Stokes flows with an arbitrary chemical reaction occurring on the particle surface at the rate $K_s F_s(C_*)$ (C_* is the concentration). It was shown [29, 30] that in these cases, at small Peclet numbers, the mean Sherwood number was determined by solving the following algebraic (transcendental) equation (C_∞ is the unperturbed concentration at infinity)

$$Sh = k_s f_s\left(\frac{Sh}{Sh_\partial}\right)\left[k_s f_s(x) \equiv \frac{aK_s}{DC_\infty} F_s(C_\infty(1-x)\right], \quad (58)$$

where the auxiliary Sherwood number Sh_∂ corresponded to purely diffusional reaction mode on the sphere surface, which in the case of a translational flow was determined from expression (47) at $Sh_0 = 1$, $(\mathbf{f}\cdot\mathbf{i}) = 1$, and the case of an arbitrary linear shear flow had the form

$$Sh_\partial = 1+\alpha Pe^{1/2}+\alpha^2 Pe+\alpha^3 Pe^{3/2}, \quad \alpha = \alpha(G_{ij}). \quad (59)$$

The general expression to determine the numerical coefficient $\alpha = \alpha(G_{ij})$ is given in ref. [14]; thus, for simple shear $\alpha = 0.257$ [12].

If now it will be 'forgotten' that the parameter Sh_∂ in expression (58) is prescribed by asymptotic expansions (47) and (59) and it will be determined directly by solving the full problem on mass transfer of a diffusively reacting sphere at finite Peclet numbers, then the approximate formula (58) can be used at intermediate and even large Peclet numbers.

A corresponding comparison [30] (see also [5]) of formula (58) with the available numerical results shows that the maximum error of interpolation formula (58) for $F_s = C_*^n (n = 1/2, 1, 2)$ is 6–9% within the whole range of Peclet numbers, $0 \leqslant Pe < \infty$.

It should be noted that in the case of a translational and shear flow past a solid sphere the parameter $Sh_\partial = Sh_\partial(Pe)$ figuring in expression (58) can be calculated by using formula (7) and (11), respectively.

6.5. *Convective heat transfer of a particle in the case of an arbitrary temperature dependence of the heat conduction coefficient*

Now, consider the convective heat exchange of an arbitrarily shaped particle with a flow in the case of an arbitrary temperature dependence of the liquid heat conduction coefficient, $\lambda_* = \lambda_*(T_*)$.

The corresponding non-linear boundary-value problem has the form

$$Pe(\mathbf{v}\cdot\nabla)T = \mathrm{div}\,(\lambda\nabla T); \quad r = r_s(\theta,\varphi),$$

$$T = 1; \quad r\to\infty, T\to 0 \quad (60)$$

$$Pe = \frac{aUc_p\rho}{\lambda_*(T_\infty)}, \quad \lambda = \lambda(T) = \frac{\lambda_*(T_*)}{\lambda_*(T_\infty)}, \quad T = \frac{T_\infty - T_*}{T_\infty - T_s},$$

where ρ and c_p is the density and specific heat of liquid, respectively, U is the characteristic flow velocity which for a translational flow is given by the quantity U_∞ and in the case of a shear flow, by a^2G; as always it was assumed for relations (60) that the condition $c_p\rho = \mathrm{const.}$, is fulfilled.

It was shown in work [23] that at small Peclet numbers, the mean Nusselt number, corresponding to the solution of problem (60), is determined by the formula

$$Nu(\lambda, Pe) = \langle\lambda\rangle Nu(1, Pe), \quad \langle\lambda\rangle = \int_0^1 \lambda(T)\,\mathrm{d}T, \quad (61)$$

where the auxiliary Nusselt number $Nu(1, Pe)$ can be found from the solution of an analogous linear problem (60) at $\lambda = 1$ ($\lambda_* = \mathrm{const.}$) and is given by the expansions of the type (47) and (59).

Formula (61) can presumably also be used for an approximate calculation of the mean Nusselt number $Nu(\lambda, Pe)$ in a much more wider range of Peclet numbers if the quantity $Nu(1, Pe)$ is determined by the solution of the linear auxiliary problem (60) at $\lambda = 1$ and $Pe = 0(1)$.

It can be shown that for a spherical particle freely suspended in a simple linear shear flow, approximate formula (61) turns to be asymptotically exact also in the other limiting case, at $Pe = \infty$. This provides the basis for the assumption that in this case expression (61) allows an approximate calculation of the mean Nusselt number at all Peclet numbers, $0 \leqslant Pe \leqslant \infty$; in this case the parameter $Nu(1, Pe)$ can, in particular, be calculated by formula (9), where Sh should be replaced by $Nu(1, Pe)$.

Summing up what has been said already, it should be noted that the comparison of the results obtained in this work with a variety of specific characteristic cases, for which the exact or approximate formulae were available, shows both good accuracy and the wide possibilities of the proposed methods of asymptotic interpolation and extrapolation.

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NOUVELLES METHODES DANS LA THEORIE DU TRANSFERT DE CHALEUR ET DE MASSE. II—LES METHODES D'INTERPOLATION ET D'EXTRAPOLATION ASYMPTOTIQUES

Résumé—Une méthode de construction de formules approchées, basée sur l'utilisation d'expressions asymptotiques sur deux côtés de la solution du problème correspondant (méthode d'interpolation asymptotique) est suggérée avec la technique d'extension du domaine de validité d'une formule asymptotique sur un côté, basée sur la procédure d'élimination de petit ou grand paramètre (méthode d'extrapolation asymptotique). Ceci est présenté sur un exemple de transfert de chaleur et de masse et des problèmes d'hydrodynamique. Une comparaison des résultats obtenus dans une variété de cas spécifiques dont les formules exactes et approchées sont connues, montre une bonne précision et les larges possibilités de la méthode présentée.

NEUE VERFAHREN IN DER THEORIE DES WÄRME- UND STOFFAUSTAUSCHES—II. DAS VERFAHREN DER ASYMPTOTISCHEN INTERPOLATION UND EXTRAPOLATION

Zusammenfassung—Es wird ein Verfahren zur Erstellung von Näherungsformeln betrachtet, das auf der Anwendung der bekannten zweiseitig asymptotischen Lösungsform auf das entsprechende Problem (die Methode der asymptotischen Interpolation) beruht. Außerdem wird auf die Technik zur substantiellen Ausweitung des Gültigkeitsbereichs von einseitig asymptotischen Formeln eingegangen, die auf dem Verfahren der "Elimination" der kleinen oder großen Parameter (das Verfahren der asymptotischen Extrapolation) basiert. Der Stoff wird anhand eines Beispiels des konvektiven Stoff- und Wärmeübergangs sowie strömungsmechanischer Probleme erläutert. Ein Vergleich der Ergebnisse aus einer Vielzahl spezieller Anwendungsfälle, für die exakte Ausdrücke und Näherungsformeln verfügbar sind, zeigt eine gute Übereinstimmung und spricht für weite Anwendungsmöglichkeiten der vorgeschlagenen Verfahren.

НОВЫЕ МЕТОДЫ ТЕОРИИ МАССО-И ТЕПЛОПЕРЕНОСА—II. МЕТОДЫ АСИМПТОТИЧЕСКОЙ ИНТЕРПОЛЯЦИИ И ЭКСТРАПОЛЯЦИИ

Аннотация—В работе предлагается метод построения приближенных формул, основанный на использовании известных двусторонних асимптотик решения соответствующей задачи (метод асимптотической интерполяции), а также способ существенного расширения диапазона применимости односторонних асимптотических формул, базирующийся на процедуре «исключения» малого или большого параметра (метод асимптотической экстраполяции). Изложение ведется на примере задач конвективного массо-и теплообмена и гидродинамики. Сопоставление полученных результатов с целым рядом частных характерных случаев, для которых уже имеются необходимые для проверки точные или приближенные формулы, показывает хорошую точность и широкие возможности предложенных методов.